

ON THE NUMBER OF FACTORIZATIONS OF AN INTEGER

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ABSTRACT. Let $f(n)$ denote the number of unordered factorizations of a positive integer n into factors larger than 1. We show that the number of distinct values of $f(n)$, less than or equal to x , is at most $\exp\left(C\sqrt{\frac{\log x}{\log \log x}}(1+o(1))\right)$, where $C = 2\pi\sqrt{2/3}$ and x is sufficiently large. This improves upon a previous result of the first author and F. Luca.

1. INTRODUCTION

Let $f(n)$ denote the number of unordered factorizations of n into factors larger than 1. More precisely, $f(n)$ is the number of tuples (n_1, \dots, n_r) , such that $1 < n_1 \leq n_2 \leq \dots \leq n_r$ and $n = n_1 n_2 \dots n_r$. For example, $f(18) = 4$, since 18 has the factorizations

$$18, \quad 2 \cdot 9, \quad 3 \cdot 6, \quad 2 \cdot 3 \cdot 3.$$

The function $f(n)$ is a multiplicative analogue of the the partition function.

There are various results on the properties of this function. The problem of determining the exact nature of $f(n)$ was considered by Oppenheim [Opp]. He proved that

$$\sum_{n \leq x} f(n) \sim \frac{x \exp(2\sqrt{\log x})}{2\sqrt{\pi}(\log x)^{3/4}}. \quad (1.1)$$

Further investigation was carried out by E.R. Canfield, P. Erdős and C. Pomerance [CEP], who showed that the maximal order of $f(n)$ is

$$n \exp\left((-1+o(1))\frac{l_1(n)l_3(n)}{l_2(n)}\right), \quad (1.2)$$

where $l_k(n)$ is the k -fold iteration of the natural logarithm.

Definition 1.1. For any $x \geq 1$, let $\mathcal{F}(x)$ be the the set of values of $f(n)$, not exceeding x , i.e.

$$\mathcal{F}(x) = \{f(n) : f(n) \leq x\}. \quad (1.3)$$

In [CEP], the authors claimed that they could prove $\#\mathcal{F}(x) = x^{o(1)}$, as $x \rightarrow \infty$. In this connection, F. Luca, A. Mukhopadhyay and K. Srinivas [LMS], proved that

$$\#\mathcal{F}(x) = x^{O(\log \log \log x / \log \log x)}. \quad (1.4)$$

This bound was improved in [BL] by the first author and F. Luca. They proved

$$\#\mathcal{F}(x) \leq \exp\left(9(\log x)^{2/3}\right), \quad \text{for all } x \geq 1. \quad (1.5)$$

In this paper, we further improve the above result. We prove

Theorem 1.2. Let $C = 2\pi\sqrt{2/3}$ and x be sufficiently large. Then

$$\#\mathcal{F}(x) \leq \exp\left(C\sqrt{\frac{\log x}{\log \log x}}\left(1+O\left(\frac{\log \log \log x}{\log \log x}\right)\right)\right).$$

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We have strong reasons to believe that up to a constant, the above bound for $\log \#\mathcal{F}(x)$ is the best possible. We will discuss more on this in the final section.

2. OUTLINE OF THE PROOF

In [Opp] and [CEP], the following observations were made:

- (1) For any prime q ,

$$f(q^n) = p(n), \quad (2.1)$$

where $p(n)$ is the partition function.

- (2) If p_1, p_2, \dots, p_r are distinct primes, then

$$f(p_1 \dots p_r) = B_r, \quad (2.2)$$

where B_r is the r^{th} Bell number, which is also the number of partitions of a set having r distinct elements.

In view of these observations, we define a generalization of the partition function to the elements of \mathbb{N}^r .

Notation. For any $r \geq 1$, let

$$\mathbb{Z}^+(r) := (\mathbb{Z}_{\geq 0})^r \setminus \{\mathbf{0}\}, \quad \text{where } \mathbf{0} = (0, \dots, 0). \quad (2.3)$$

Definition 2.1. Let $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$. A partition of α is an unordered decomposition

$$\alpha = \beta_1 + \dots + \beta_l,$$

where $\beta_i \in \mathbb{Z}^+(r)$, for each $1 \leq i \leq l$ and the addition is component-wise. The number of partitions of α is denoted by $p(\alpha)$.

Example 2.2. The partitions of $\alpha = (1, 2)$ are

$$(1, 2), \quad (1, 0) + (0, 2), \quad (0, 1) + (1, 1), \quad (0, 1) + (0, 1) + (1, 0).$$

Remark 2.3. When $r = 1$, the above corresponds to the usual *partition function* in \mathbb{N} . Moreover, any such partition π of $\alpha \in \mathbb{N}^r$ can be represented as

$$\pi = \prod_{\beta \in \mathbb{Z}^+(r)} \beta^{\pi(\beta)},$$

as in the case $r = 1$.

Remark 2.4. The above function can also be thought of as a partition of the multi-set

$$\{1, 1, \dots, 1, 2, \dots, 2, \dots, r, \dots, r\},$$

with each i having exactly α_i copies, for $1 \leq i \leq r$. When $\alpha_i = 1$ for each i , this corresponds to a set-partition, the number of which is given by the r^{th} Bell number B_r .

The following lemma generalizes the observations in (2.1) and (2.2).

Lemma 2.5. Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ and $\alpha = (\alpha_1, \dots, \alpha_r)$. Then

$$f(n) = p(\alpha).$$

Proof. Let $n = n_1 n_2 \dots n_l$ be a nontrivial factorization of n , with $n_i > 1$ for each i . For each $1 \leq i \leq l$, let

$$n_i = \prod_{j=1}^r p_j^{\beta_{ij}} \quad \text{and} \quad \beta_i = (\beta_{i1}, \dots, \beta_{ir}).$$

Then, clearly $\beta_i \in \mathbb{Z}^+(r)$ and $\sum_{i=1}^l \beta_i = \alpha$. Therefore, each unordered factorization gives rise to a partition of α . Clearly, the partition obtained in this way is unique. The converse follows analogously. \square

Therefore, $\#\mathcal{F}(x)$ is bounded above by the number of unordered tuples $\alpha = (\alpha_1, \dots, \alpha_r)$, which satisfy $p(\alpha) \leq x$. We record this as the following Corollary:

Corollary 2.6.

$$\#\mathcal{F}(x) \leq \#\{1 \leq \alpha_1 \leq \dots \leq \alpha_r : p(\alpha) \leq x\}.$$

The problem has now reduced to determining the distribution of $p(\alpha) \leq x$. Therefore, we seek a lower bound for $p(\alpha)$.

Proposition 2.7. *Let $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$. For any $z > 0$, let*

$$g(\alpha, z) = z \prod_{i=1}^r \left(1 + \frac{\alpha_i}{z}\right)^{-1}. \quad (2.4)$$

Then $g(\alpha, z)$ is a strictly increasing function whose value at 1 is less than 1. Let $z(\alpha) > 1$ be the unique positive real solution to the equation $g(\alpha, z) = 1$ and let $N = N(\alpha)$, be the greatest integer less than or equal to $z(\alpha)$, i.e., $N = \lfloor z(\alpha) \rfloor \geq 1$. Then

(a)

$$p(\alpha) \geq \frac{e^{N-2}}{2N^{\frac{3}{2}}} \prod_{i=1}^r \frac{1}{2\sqrt{2N}} \left(1 + \frac{N}{\alpha_i}\right)^{\alpha_i + \frac{1}{2}}.$$

(b) *Further, if $p(\alpha) \leq x$, then for x sufficiently large, we have*

$$r \leq R = \frac{2 \log x}{\log \log x} \left(1 + \frac{2 \log \log \log x}{\log \log x}\right) \quad \text{and} \quad N \leq 3 \log x.$$

Notation. The quantity $N = N(\alpha)$ depends entirely on α . For sake of simplicity, we write this as N .

We now prove Theorem 1.2 using Proposition 2.7. We assume throughout, that x is sufficiently large.

Let $\alpha \in \mathbb{N}^r$ be such that $p(\alpha) \leq x$. Taking logarithm in the inequality in Proposition 2.7 (a), and transferring the negative terms to RHS, we obtain

$$N + \sum_{i=1}^r (\alpha_i + 0.5) \log \left(1 + \frac{N}{\alpha_i}\right) \leq \log x + 0.5(r + 3) \log N + 1.04r + 2.7.$$

Using the bounds for N and r from Proposition 2.7 (b) in the RHS above, and simplifying, we get

$$\sum_{i=1}^r \alpha_i \log \left(1 + \frac{N}{\alpha_i}\right) \leq 2 \log x \left(1 + O\left(\frac{\log \log \log x}{\log \log x}\right)\right). \quad (2.5)$$

Next, we split the set $\{\alpha_1, \dots, \alpha_r\}$ into two parts I and J , where

$$I = \{\alpha_i : \alpha_i \leq A(N + 1)\} \quad \text{and} \quad J = \{\alpha_i : \alpha_i > A(N + 1)\},$$

and $A > 0$ is a positive constant. We shall choose

$$A = \frac{(\log \log x)^6}{(\log x)^{1/2}}. \quad (2.6)$$

We separately estimate the number of choices for elements in I and J .

For elements of I , we have $\alpha_i \leq A(N + 1)$. Therefore, it follows that

$$\log \left(1 + \frac{N}{\alpha_i}\right) \geq \log \left(1 + \frac{N}{A(N + 1)}\right) \geq \log \left(1 + \frac{1}{2A}\right) \geq \frac{\log \log x}{2} \left(1 + O\left(\frac{\log \log \log x}{\log \log x}\right)\right).$$

for all $\alpha_i \in I$. With this applied to (2.5), we obtain (ignoring the elements of J)

$$\sum_I \alpha_i \leq \frac{4 \log x}{\log \log x} \left(1 + O \left(\frac{\log \log \log x}{\log \log x} \right) \right). \quad (2.7)$$

The following lemma gives us the required upper bound for the number of such α_i .

Lemma 2.8. *The number of unordered tuples (n_1, \dots, n_l) of positive integers, for which*

$$\sum_{i=1}^l n_i \leq y,$$

is at most $y \exp \left(\pi \sqrt{2y/3} \right)$, for all $y \geq 1$.

Remark 2.9. The bound for the number of solutions above is actually $O(\sqrt{y} \exp(\pi \sqrt{2y/3}))$. As this is not quite useful for us, we keep the bound as above to make the proof easier.

Proof of Lemma 2.8. Suppose that $\sum_{i=1}^l n_i = n \leq y$. From the proof of Theorem 15.3 in [Nat, Pg 468], we have the upper bound

$$p(n) \leq \exp \left(\pi \sqrt{2n/3} \right), \quad \text{for all } n \geq 1.$$

Therefore, the total number of choices for n_1, \dots, n_l is at most

$$\sum_{n \leq y} \exp \left(\pi \sqrt{2n/3} \right) \leq y \exp \left(\pi \sqrt{2y/3} \right).$$

□

Applying Lemma 2.8 to (2.7), the total number of choices for α_i 's in I , is at most

$$\exp \left(2\pi \sqrt{\frac{2 \log x}{3 \log \log x}} \left(1 + O \left(\frac{\log \log \log x}{\log \log x} \right) \right) \right). \quad (2.8)$$

Next, we estimate the total number of choices for elements of J . Observe that for any $1 \leq i \leq r$, we have $p(\alpha_i) \leq p(\alpha) \leq x$. Moreover, from Corollary 3.1 of [Mar], we also have the lower bound

$$p(n) \geq \frac{\exp(2\sqrt{n})}{14}, \quad \text{for all } n \geq 1.$$

Therefore, in particular, for each $\alpha_i \in J$, we have

$$\alpha_i \leq \frac{1}{4} (\log 14x)^2 \leq \log^2 x, \quad (2.9)$$

In the next lemma, we estimate the cardinality of J .

Lemma 2.10. *With J as before, we have*

$$\#J \leq \frac{4\sqrt{\log x}}{(\log \log x)^5}.$$

Proof. Note that $g(\alpha, z)$ is strictly increasing by Proposition 2.7, with $z(\alpha)$ being the unique positive real solution to $g(\alpha, z) = 1$. As $N \leq z(\alpha) \leq N+1$, we have $g(\alpha, N+1) \geq 1$. Therefore

$$N+1 \geq \prod_{i=1}^r \left(1 + \frac{\alpha_i}{N+1} \right) \geq \prod_{\alpha_i \in J} \left(1 + \frac{\alpha_i}{N+1} \right) \geq (1+A)^{\#J},$$

since $\alpha_i > A(N+1)$, for all $\alpha_i \in J$.

Since $A < 1$, we have $\log(1 + A) \geq A/2$ and from Proposition 2.7, we have $\log(N + 1) \leq \log(1 + 3 \log x) \leq 2 \log \log x$. Hence

$$\#J \leq \frac{\log(N + 1)}{\log(A + 1)} \leq \frac{4\sqrt{\log x}}{(\log \log x)^5}.$$

This proves the lemma. \square

From (2.9) and Lemma 2.10, the number of choices for elements of J is at most

$$(\log^2 x)^{\#J} \leq \exp\left(\frac{8\sqrt{\log x}}{(\log \log x)^4}\right), \quad (2.10)$$

Therefore, from (2.8) and (2.10), the total number of choices for α is at most

$$\exp\left(2\pi\sqrt{2/3}\sqrt{\frac{\log x}{\log \log x}}\left(1 + O\left(\frac{\log \log \log x}{\log \log x}\right)\right)\right).$$

This completes the proof of Theorem 1.2.

It now remains to give a proof of Proposition 2.7.

3. PRELIMINARY LEMMAS

In this section, we prove some Preliminary results.

3.1. Bounds on factorials and binomials. We begin with the following lemma.

Lemma 3.1. *Let*

$$h_1(x) = \left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}}, \quad h_2(x) = \frac{x+1}{x+2} \left(1 + \frac{1}{x}\right)^{x+\frac{3}{2}}.$$

Then, as $x \rightarrow \infty$, the functions h_1 and h_2 converge to e decreasingly.

Next, we obtain bounds for factorials and binomial coefficients.

Lemma 3.2. *Let n and k be positive integers. Then*

(a)

$$(k+1)! \leq \frac{2k^{k+\frac{3}{2}}}{e^{k-1}},$$

(b)

$$\binom{k+n}{k} \geq \frac{1}{2\sqrt{2}} \frac{(k+n)^{k+n+\frac{1}{2}}}{k^{k+\frac{1}{2}}n^{n+\frac{1}{2}}}.$$

Proof. Proof is by induction on k . We first prove (a).

When $k = 1$, (a) is trivially true. So, assume that (a) holds for some $k \geq 1$. Then, by induction

$$(k+2)! = (k+2)(k+1)! \leq \frac{2(k+2)k^{k+\frac{3}{2}}}{e^{k-1}}. \quad (3.1)$$

We need to show that the RHS of (3.1) is at most

$$\frac{2(k+1)^{k+\frac{5}{2}}}{e^k},$$

which is equivalent to

$$\frac{k+1}{k+2} \left(1 + \frac{1}{k}\right)^{k+\frac{3}{2}} \geq e,$$

and this is true by Lemma 3.1 for the function h_2 .

Next, we prove (b). When $k = 1$, this reduces to

$$\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} \leq 2\sqrt{2}.$$

This is true from Lemma 3.1, since the function h_1 is decreasing and therefore its maximum on the positive integers is attained at $n = 1$.

Now, suppose that the (b) holds true for (k, n) . Then, by induction

$$\binom{k+n+1}{k+1} = \frac{k+n+1}{k+1} \binom{k+n}{k} \geq \frac{1}{2\sqrt{2}} \frac{(k+n+1)}{(k+1)} \frac{(k+n)^{k+n+\frac{1}{2}}}{k^{k+\frac{1}{2}} n^{n+\frac{1}{2}}}. \quad (3.2)$$

We need to show that the RHS of (3.2) is at least

$$\frac{1}{2\sqrt{2}} \frac{(k+n+1)^{k+n+\frac{3}{2}}}{(k+1)^{k+\frac{3}{2}} n^{n+\frac{1}{2}}}.$$

This is equivalent to

$$\left(1 + \frac{1}{k}\right)^{k+\frac{1}{2}} \geq \left(1 + \frac{1}{k+n}\right)^{k+n+\frac{1}{2}},$$

which is true since h_1 is decreasing from Lemma 3.1. This completes the proof. \square

3.2. A generating function for $p(\alpha)$. We give a generating function for $p(\alpha)$, which we later use to obtain a lower bound for $p(\alpha)$. We use the following notation:

Notation. Let $\mathbf{q} = (q_1, \dots, q_r)$, with $|q_i| < 1$ for each $1 \leq i \leq r$. For $\beta \in \mathbb{Z}^+(r)$, we use the notation

$$\mathbf{q}^\beta := q_1^{\beta_1} \dots q_r^{\beta_r}.$$

We have

Lemma 3.3. *Let*

$$P(\mathbf{q}) = \prod_{\beta \in \mathbb{Z}^+(r)} (1 - \mathbf{q}^\beta)^{-1}.$$

Then $P(\mathbf{q})$ is a generating function for $p(\alpha)$ i.e., for any $\alpha \in \mathbb{N}^r$, the coefficient of \mathbf{q}^α in $P(\mathbf{q})$ is $p(\alpha)$.

Remark 3.4. When $r = 1$, the above corresponds to the generating function of the partition function $p(n)$.

Proof of Lemma 3.3. Since the given product converges locally uniformly, we can write it as

$$P(\mathbf{q}) = \prod_{\beta \in \mathbb{Z}^+(r)} \left(\sum_{l=0}^{\infty} \mathbf{q}^{l\beta} \right) = \sum_{h: \mathbb{Z}^+(r) \rightarrow \mathbb{Z}_{\geq 0}} \mathbf{q}^{h(\beta) \cdot \beta} \quad (3.3)$$

Therefore, the coefficient of \mathbf{q}^α above equals the number of all functions $h: \mathbb{Z}^+(r) \rightarrow \mathbb{Z}_{\geq 0}$, for which

$$\sum_{\beta \in \mathbb{Z}^+(r)} h(\beta) \cdot \beta = \alpha.$$

We show that the above quantity equals $p(\alpha)$. Suppose that π is a partition of α . Then one can write π as

$$\pi = \prod_{\beta \in \mathbb{Z}^+(r)} \beta^{h(\beta)}.$$

Clearly, the above gives rise to a unique such function h . Conversely, any such function h gives a unique product decomposition as above. This completes the proof. \square

We prove the following lemma about the exponential of a power series:

Lemma 3.5. *Suppose that*

$$F(\mathbf{q}) = a(\mathbf{0}) + \sum_{\mathbf{n} \in \mathbb{Z}^+(r)} a(\mathbf{n}) \mathbf{q}^{\mathbf{n}},$$

is convergent in $\{\mathbf{q} : |q_i| < 1\}$, with real coefficients satisfying $a(\mathbf{n}) \geq 0$, for $\mathbf{n} \in \mathbb{Z}^+(r) \cup \{\mathbf{0}\}$. Then the power series of $G(\mathbf{q}) = \exp(F(\mathbf{q}))$ around $\mathbf{0}$ also has non-negative coefficients.

Proof. Note that

$$G(\mathbf{q}) = \sum_{k=0}^{\infty} \frac{F(\mathbf{q})^k}{k!}.$$

Now, since $a(\mathbf{n}) \geq 0$, for each $\mathbf{n} \in \mathbb{Z}^+(r)$, it follows that the coefficients of $F(\mathbf{q})^k$ are non-negative for each $k \geq 0$. Therefore, $G(\mathbf{q})$ has non-negative coefficients. \square

Next, we obtain a lower bound for $p(\boldsymbol{\alpha})$.

Lemma 3.6. *Let $\boldsymbol{\alpha} \in \mathbb{N}^r$. Then*

$$p(\boldsymbol{\alpha}) \geq \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \prod_{i=1}^r \binom{k + \alpha_i}{k}. \quad (3.4)$$

Remark 3.7. The RHS of (3.4) can be written in terms of a generalized hypergeometric series as

$$\frac{1}{e} {}_rF_r \left(\begin{matrix} \alpha_1 + 1 & \dots & \dots & \alpha_{r-1} + 1 & \alpha_r + 1 \\ 1 & \dots & \dots & 1 & 2 \end{matrix} ; 1 \right).$$

When $\boldsymbol{\alpha} = (1, 1, \dots, 1)$, equality holds in (3.4) and the RHS of (3.4) becomes the Dobiński's formula for the r^{th} Bell number B_r .

Proof of Lemma 3.6. Taking logarithms in the expression for $P(\mathbf{q})$ in Lemma 3.3, we get

$$\begin{aligned} \log P(\mathbf{q}) &= \sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} -\log(1 - \mathbf{q}^{\boldsymbol{\beta}}) = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \sum_{m=1}^{\infty} \frac{\mathbf{q}^{m\boldsymbol{\beta}}}{m} = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \mathbf{q}^{\boldsymbol{\beta}} \sum_{m|\boldsymbol{\beta}_i \forall i} \frac{1}{m} \\ &= \sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \frac{\sigma(\beta_1, \dots, \beta_r)}{(\beta_1, \dots, \beta_r)} \mathbf{q}^{\boldsymbol{\beta}} \\ &= \sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \mathbf{q}^{\boldsymbol{\beta}} + H(\mathbf{q}), \end{aligned} \quad (3.5)$$

where $\sigma(\beta_1, \dots, \beta_r)$ denotes $\sigma(\gcd(\beta_1, \dots, \beta_r))$, and

$$H(\mathbf{q}) = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \left(\frac{\sigma(\beta_1, \dots, \beta_r)}{(\beta_1, \dots, \beta_r)} - 1 \right) \mathbf{q}^{\boldsymbol{\beta}}. \quad (3.6)$$

Taking exponential in (3.5), we get

$$P(\mathbf{q}) = \exp \left(\sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \mathbf{q}^{\boldsymbol{\beta}} \right) \cdot \exp(H(\mathbf{q})). \quad (3.7)$$

Now, we have

$$\sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \mathbf{q}^{\boldsymbol{\beta}} = \sum_{\beta_i, \dots, \beta_r \geq 0} q_1^{\beta_1} \dots q_r^{\beta_r} - 1 = \frac{1}{(1 - q_1) \dots (1 - q_r)} - 1. \quad (3.8)$$

Note that $H(\mathbf{q})$ has non-negative coefficients with constant term 0. Therefore, by Lemma 3.5, $\exp(H(\mathbf{q}))$ also has non-negative coefficients with constant term 1. Therefore, the coefficient of $\mathbf{q}^{\boldsymbol{\alpha}}$ in $P(\mathbf{q})$ is at least $1/e$ times the coefficient of $\mathbf{q}^{\boldsymbol{\alpha}}$ in $\exp \left(\prod_{i=1}^r (1 - q_i)^{-1} \right)$.

Since

$$\exp\left(\prod_{i=1}^r(1-q_i)^{-1}\right) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^r(1-q_i)^{-k}, \quad (3.9)$$

and

$$(1-q)^{-k} = 1 + \sum_{n=1}^{\infty} \binom{k+n-1}{k-1} q^n,$$

the coefficient of q^α in (3.9) equals

$$\sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^r \binom{k+\alpha_i-1}{k-1} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \prod_{i=1}^r \binom{k+\alpha_i}{k}.$$

This completes the proof. \square

We are now in a position to give a proof of Proposition 2.7.

4. PROOF OF PROPOSITION 2.7

Firstly, we have

$$g(\alpha, z) = z \prod_{i=1}^r \left(1 + \frac{\alpha_i}{z}\right)^{-1}.$$

Taking logarithmic derivative, we find that

$$\frac{g'(\alpha, z)}{g(\alpha, z)} = \frac{r+1}{z} - \sum_{i=1}^r \frac{1}{z+\alpha_i} > 0,$$

for all $z > 0$.

Therefore, $g(\alpha, z)$ is a strictly increasing function in z with $g(\alpha, 1) < 1$. Hence, the equation $g(\alpha, z) = 1$ must have a unique positive real solution $z(\alpha) > 1$. Therefore, with $N = \lfloor z(\alpha) \rfloor \geq 1$, one has

$$g(\alpha, N) \leq 1 \leq g(\alpha, N+1). \quad (4.1)$$

In particular, we have

$$\prod_{i=1}^r \left(1 + \frac{\alpha_i}{N}\right) \geq N. \quad (4.2)$$

We now prove (a). We will use the bound given by a hypergeometric series for $p(\alpha)$ from Lemma 3.6, namely

$$p(\alpha) \geq \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \prod_{i=1}^r \binom{k+\alpha_i}{k} = \frac{1}{e} \sum_{k=0}^{\infty} T(\alpha, k). \quad (4.3)$$

We do not have an asymptotic formula for this sum. Fortunately for us, the hypergeometric series converges quite rapidly and therefore only one term $T(\alpha, k)$ will be good enough to give a decent lower bound, provided k is optimally chosen.

Applying Lemma 3.2 to $T(\alpha, k)$, we have for any $k \geq 1$, that

$$T(\alpha, k) \geq \frac{e^{k-1}}{2k^{k+\frac{3}{2}}} \prod_{i=1}^r \frac{1}{2\sqrt{2}} \frac{(k+\alpha_i)^{k+\alpha_i+\frac{1}{2}}}{\alpha_i^{\alpha_i+\frac{1}{2}} k^{k+\frac{1}{2}}} \quad (4.4)$$

We make the choice $k = N$ in (4.4), to obtain

$$T(\alpha, N) \geq \frac{e^{N-1}}{2N^{N+\frac{3}{2}}} \prod_{i=1}^r \frac{1}{2\sqrt{2N}} \left(1 + \frac{\alpha_i}{N}\right)^N \left(1 + \frac{N}{\alpha_i}\right)^{\alpha_i+\frac{1}{2}}. \quad (4.5)$$

Using (4.2) in (4.5), we get

$$p(\alpha) \geq \frac{T(\alpha, N)}{e} \geq \frac{e^{N-2}}{2 N^{\frac{3}{2}}} \prod_{i=1}^r \frac{1}{2\sqrt{2N}} \left(1 + \frac{N}{\alpha_i}\right)^{\alpha_i + \frac{1}{2}}.$$

This proves (a).

We now prove (b). From Lemma 3.6, we have

$$p(\alpha) \geq \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \prod_{i=1}^r \binom{k + \alpha_i}{k} \geq \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^r}{k!}. \quad (4.6)$$

Taking the term $k = \lceil r/2 \rceil$, and using the inequality

$$\frac{1}{k!} \geq \frac{1}{k^k}, \quad \text{for all } k \geq 1,$$

we obtain

$$x \geq p(\alpha) \geq \frac{1}{e} \frac{\lceil r/2 \rceil^r}{\lceil r/2 \rceil!} \geq \frac{1}{e} \lceil r/2 \rceil^{\lceil r/2 \rceil}.$$

From this, it follows that $r \leq R$.

To show $N \leq 3 \log x$, we take logarithms in (a) of Proposition 2.7, to get

$$N - 1.04R - 0.5(R+3) \log N - \log x - 2.7 \leq 0.$$

Substituting R , it follows that $N \leq 3 \log x$. This completes the proof of Proposition 2.7.

5. CONCLUDING REMARKS

We believe that the bound in Theorem 1.2 is essentially the best possible due to the following reasons. Let

$$S = \left\{ \alpha : \alpha_i \leq \sqrt{\log x} \ \forall i, \ \sum \alpha_i \leq \frac{B \log x}{\log \log x} \right\}.$$

Then, for each $\alpha \in S$, we have $p(\alpha) = O(x)$. Moreover, the number of elements in this set is at least $\exp\left(c_1 \sqrt{\frac{\log x}{\log \log x}}\right)$. But we are not able to show that the values of $p(\alpha)$, as α runs through S , are distinct. However, some calculations seem to show that the number of distinct values of $p(\alpha)$ above are also having a similar lower bound. We shall return to this later.

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